

SHARP SPECTRAL ESTIMATES FOR PERIODIC MATRIX-VALUED JACOBI OPERATORS

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ABSTRACT. For the periodic matrix-valued Jacobi operator J we obtain the estimate of the Lebesgue measure of the spectrum $|\sigma(J)| \leq 4 \min_n \text{Tr}(a_n a_n^*)^{\frac{1}{2}}$, where a_n are off-diagonal elements of J . Moreover estimates of width of spectral bands are obtained.

1. INTRODUCTION

We consider a self-adjoint matrix-valued Jacobi operator $J : \ell^2(\mathbb{Z})^m \rightarrow \ell^2(\mathbb{Z})^m$ given by

$$(Jy)_n = a_n y_{n+1} + b_n y_n + a_{n-1}^* y_{n-1}, \quad n \in \mathbb{Z}, \quad y_n \in \mathbb{C}^m, \quad y = (y_n)_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})^m, \quad (1.1)$$

where a_n and $b_n = b_n^*$ are p -periodic sequences of the complex $m \times m$ matrices. It is well known (see e.g. [CG], [CGR], [KKu], [KKu1]) that the spectrum of this operator $\sigma(J) = \sigma_{ac}(J) \cup \sigma_p(J)$, where absolutely continuous part $\sigma_{ac}(J)$ is a union of finite number of intervals and $\sigma_p(J)$ consists of finite number of eigenvalues of infinite multiplicity. Note that if $\det a_n \neq 0$ for all $n = 1, \dots, p$, then $\sigma(J) = \sigma_{ac}(J)$ and $\sigma_p(J) = \emptyset$ always. The main goal of this paper is to obtain estimate of length of spectrum $\sigma(J)$. We don't know such estimates for the matrix-valued Jacobi operators.

Theorem 1.1. *The Lebesgue measure of spectrum of J satisfy the following estimate*

$$\text{mes}(\sigma(J)) \leq 4 \min_n \text{Tr}(a_n a_n^*)^{\frac{1}{2}}. \quad (1.2)$$

Remark 1. Note that this estimate does not depend on period p and coefficients b_n , i.e. changing p and b_n we can't sufficiently increase the length of spectrum of J . If $a_n = 0$ for some $n \in \mathbb{Z}$, then this estimate is sharp.

2. ($m = 1$) For the scalar case $m = 1$ we have the estimate (see e.g. [DS], [Ku], [KKr])

$$\text{mes}(\sigma(J)) \leq 4 |a_1 a_2 \dots a_p|^{\frac{1}{p}}, \quad (1.3)$$

We reach equality for the case of discrete Shrödinger operator J^0 with $a_n^0 = 1$, $b_n^0 = 0$. Estimate (1.2) is better than (1.3), since $\min |a_n| \leq |a_1 a_2 \dots a_p|^{\frac{1}{p}}$.

3. The result similar to (1.3) was obtained in [PR] for general non periodic scalar case ($m = 1$).

4. From the Proof of Theorem 1.1 we obtain estimates of spectral bands, see (2.9).

Example (sharpness). We construct the Jacobi matrix J whose spectrum satisfy $\text{mes}(\sigma(J)) = 4 \min_n (\text{Tr } a_n a_n^*)^{\frac{1}{2}}$. Let J be Jacobi matrix with elements $a_n = I_m$ ($m \times m$ identical matrix) and $b_n = \text{diag}(4k)_{k=1}^m$ for any n . Since all a_n and b_n are diagonal matrix, then J is unitarily

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equivalent to the direct sum of scalar Jacobi operators. In our case this is the direct sum of shifted discrete Shrödinger operators $\oplus_{k=1}^m (J^0 + 4kI)$ (I is identical operator). Then

$$\sigma(J) = \bigcup_{k=1}^m \sigma(J^0 + 4kI) = \bigcup_{k=1}^m [-2 + 4k, 2 + 4k] = [2, 2 + 4m],$$

which gives us $\text{mes}(\sigma(J)) = 4m = 4(\text{Tr } a_n a_n^*)^{\frac{1}{2}}$.

2. PROOF OF THEOREM 1.1

Without lost of generality we may assume that $\min_n \text{Tr}(a_n a_n^*)^{\frac{1}{2}} = \text{Tr}(a_0 a_0^*)^{\frac{1}{2}}$ and $p \geq 3$. It is well known (see e.g. [KKu], [KKu1]) that J is unitarily equivalent to the operator $\mathcal{J} = \int_{[0, 2\pi)}^{\oplus} K(x) dx$ acting in $\int_{[0, 2\pi)}^{\oplus} \mathcal{H} dx$, where $\mathcal{H} = \mathbb{C}^{pm}$ and $pm \times pm$ matrix $K(x)$ is given by

$$K(x) = \begin{pmatrix} b_1 & a_1 & 0 & \dots & e^{-ix} a_0^* \\ a_1^* & b_2 & a_2 & \dots & 0 \\ 0 & a_2^* & b_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ e^{ix} a_0 & 0 & 0 & \dots & b_p \end{pmatrix} = K_0 + K_1(x), \quad (2.4)$$

where

$$K_0 = \begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 \\ a_1^* & b_2 & a_2 & \dots & 0 \\ 0 & a_2^* & b_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & b_p \end{pmatrix}, \quad K_1(x) = \begin{pmatrix} 0 & 0 & 0 & \dots & e^{-ix} a_0^* \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ e^{ix} a_0 & 0 & 0 & \dots & 0 \end{pmatrix}. \quad (2.5)$$

The spectrum $\sigma(J)$ is

$$\sigma(J) = \bigcup_{x \in [0, 2\pi]} \sigma(K(x)). \quad (2.6)$$

From (2.4)-(2.5) we obtain

$$K_0 - |K_1| \leq K(x) \leq K_0 + |K_1|, \quad x \in [0, 2\pi] \quad (2.7)$$

where

$$|K_1| = (K_1 K_1^*)^{\frac{1}{2}} = \begin{pmatrix} (a_0^* a_0)^{\frac{1}{2}} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & (a_0 a_0^*)^{\frac{1}{2}} \end{pmatrix} \quad (2.8)$$

does not depend on x . Let $\lambda_1(x) \leq \dots \leq \lambda_N(x)$ be eigenvalues of $K(x)$ and let $\lambda_1^{\pm} \leq \dots \leq \lambda_N^{\pm}$ be eigenvalues of $K_0 \pm |K_1|$. Using (2.7) we obtain

$$\lambda_n^- \leq \lambda_n(x) \leq \lambda_n^+, \quad x \in [0, 2\pi] \quad (2.9)$$

which with (2.6) gives us

$$\sigma(J) = \bigcup_{x \in [0, 2\pi]} \{\lambda_n(x)\}_{n=1}^N \subset \bigcup_{n=1}^N [\lambda_n^-, \lambda_n^+]. \quad (2.10)$$

Then

$$\text{mes}(\sigma(J)) \leq \sum_{n=1}^N (\lambda_n^+ - \lambda_n^-) = 2 \text{Tr} |K_1| = 4 \text{Tr}(a_n a_n^*) \blacksquare \quad (2.11)$$

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